Integrable Cases of Autonomic Polynomial System

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Integrability

For the autonomous ODE system

$$\frac{d\ x_i}{d\ t} = \phi_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n,$$

the full derivation of the first integral $I_k(x_1, \ldots, x_n)$ equal zero along the trajectory in phase space of that system

$$\left.\frac{d\ I_k(x_1, \ldots, x_n)}{d\ t}\right|_{\frac{d\ x_i}{d\ t} = \phi_i(x_1, \ldots, x_n)} = 0, \quad k = 1, \ldots, m.$$

- The system can have m first integrals. System is called integrable if it has enough numbers of the integrals.
- For integrability of an autonomous two-dimensional system, it is enough to have a single integral.

Example

- Let us see the harmonic oscillator

$$\ddot{x}(t) + \omega_0^2 \cdot x(t) = 0$$

- This is equivalent to the notation

$$\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = -\omega_0^2 \cdot x(t). \end{cases} \quad (1)$$

- The first integral is

$$I(x(t), y(t)) = x^2(t) + y^2(t)/\omega_0^2.$$

- Its full derivation in time is

$$\frac{dI(x(t), y(t))}{dt} = 2x(t)\dot{x}(t) + 2y(t)\dot{y}(t)/\omega_0^2 = 2x(t)y(t) - 2x(t)y(t) = 0$$

due to (1).

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Solution

Of constancy of the first integral

$$I(x(t), y(t)) = C_1^2,$$

we have

$$y(t) = \sqrt{\omega_0^2 \cdot (C_1^2 - x^2(t))}.$$

By substituting $y(t)$ in system (1) we get

$$\frac{dx(t)}{dt} = \sqrt{\omega_0^2 \cdot (C_1^2 - x^2(t))},$$

or

$$\frac{dx(t)}{\sqrt{C_1 - x^2(t)}} = \omega_0 \cdot dt, \quad \text{i.e. } \arcsin(x(t)/C_1) = \omega_0 \cdot t + C_2.$$

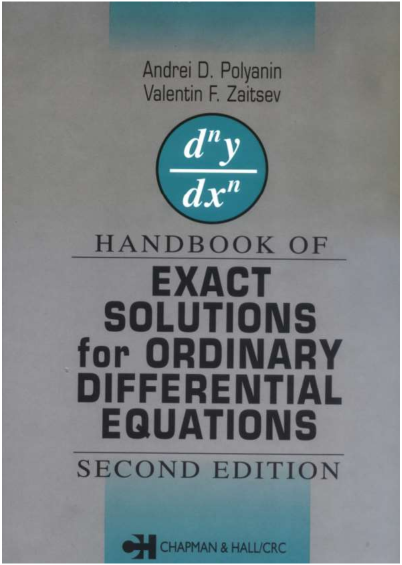
Finally we get $\quad x(t) = C_1 \cdot \sin(\omega_0 \cdot t + C_2).$

- Integrability is an important property of the system. In particular, if a system is integrable then it is solvable by quadrature.
- The knowledge of the integrals is important also at the investigation of a phase portrait, for the creation of symplectic integration schemes e.t.c.

Problem

- Generally, integrability is a rare property.
- But the system may depend on parameters.
- Our task here is to find the values of system parameters at which the system is integrable.
- To solve this problem, we try to use the local analysis. It studies the behavior of a system in a neighborhood of a point in phase space.

Polyanin, Zaitsev 2003



2.2.3-2. Solvable equations and their solutions.

1. $y''_{xx} + y + ay^3 = 0.$

Duffing equation. This is a special case of equation 2.9.1.1 with $f(y) = -y - ay^3$.

1°. Solution:

$$x = \pm \int \left(C_1 - y^2 - \tfrac{1}{2}ay^4\right)^{-1/2} dy + C_2.$$

The period of oscillations with amplitude C is expressed in terms of the complete elliptic integral of the first kind:

$$T = \frac{4}{\sqrt{1+aC^2}} K\left(\frac{aC^2}{2+2aC^2}\right), \quad \text{where} \quad K(m) = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - m\sin^2 t}}.$$

2°. The asymptotic solution, as $a \to 0$, has the form:

$$y = \widetilde{C}_1 \cos[(1 + \tfrac{3}{8}a\widetilde{C}_1^2)x + \widetilde{C}_2] + \tfrac{1}{32}a\widetilde{C}_1^3 \cos[3(1 + \tfrac{3}{8}a\widetilde{C}_1^2)x + 3\widetilde{C}_2] + O(a^2),$$

where $\widetilde{C}_1$ and $\widetilde{C}_2$ are arbitrary constants. The corresponding asymptotics for the period of oscillations with amplitude C is described by the formula: $T = 2\pi(1 - \frac{3}{8}aC^2) + O(a^2)$.

2. $y''_{xx} + ayy'_x + by^3 + cy = 0.$

The transformation $w(z) = y'_x$, $z = -\frac{1}{2}ay^2$ leads to an Abel equation of the form 1.3.1.2: $ww'_z - w = -\dfrac{2b}{a^2}z + \dfrac{c}{a}$.

3. $y''_{xx} = (ay + 3b)y'_x + cy^3 - aby^2 - 2b^2y.$

The substitution $w(y) = y'_x$ leads to an Abel equation of the form 1.3.3.1:

$$ww'_y = (ay + 3b)w + cy^3 - aby^2 - 2b^2y.$$

4. $y''_{xx} = (3ay + b)y'_x - a^2y^3 - aby^2 + cy.$

The substitution $w(y) = y'_x$ leads to an Abel equation of the form 1.3.3.2:

$$ww'_y = (3ay + b)w - a^2y^3 - aby^2 + cy.$$

5. $2y''_{xx} = (7ay + 5b)y'_x - 3a^2y^3 - 2cy^2 - 3b^2y.$

The substitution $w(y) = y'_x$ leads to an Abel equation of the form 1.3.3.3:

$$2ww'_y = (7ay + 5b)w - 3a^2y^3 - 2cy^2 - 3b^2y.$$

Local integrability

We consider an autonomous system of ordinary differential equations

$$dx_i/dt \stackrel{\text{def}}{=} \dot{x}_i = \phi_i(X), \quad i = 1, \dots, n, \tag{2}$$

where $X = (x_1, \dots, x_n) \in \mathbb{C}^n$ and $\phi_i(X)$ are polynomials.

In a neighborhood of the point $X = X^0$, the system (2) is *locally integrable* if it has there sufficient number m of independent first integrals of the form

$$I_k(X) = a_k(X)/b_k(X), \quad k = 1, \dots, m,$$

where functions $a_k(X)$ and $b_k(X)$ are analytic in a neighborhood of this point. Such $I_k(X)$ are called the formal integral.

Otherwise we call the system (2) *locally nonintegrable* in this neighborhood.

Resonance normal form

- The resonance normal form was introduced by Poincaré for the investigation of systems of nonlinear ordinary differential equations. It is based on the maximal simplification of the right-hand sides of these equations by invertible transformations.
- The normal form approach was developed in works of G.D. Birkhoff, T.M. Cherry, A. Deprit, F.G. Gustavson, C.L. Siegel, J. Moser, A.D. Bruno and others. This technique is based on the Local Analysis method by Prof. Bruno [Bruno 1971, 1972, 1979, 1989].

Multi-index notation

Let's suppose that we treat the reduced to a diagonal polynomial system near a stationary point at the origin and rewrite this n-dimension system in the terms

$$\dot{x}_i = \lambda_i x_i + x_i \sum_{\mathbf{q}\in\mathcal{N}_i} f_{i,\mathbf{q}} \mathbf{x}^{\mathbf{q}}, \quad i = 1, \ldots, n\,, \tag{3}$$

where we use the multi-index notation

$$\mathbf{x}^{\mathbf{q}} \equiv \prod_{j=1}^{n} {x_j}^{q_j}$$

with the power exponent vector $\mathbf{q} = (q_1, \ldots, q_n)$
Here the sets:

$$\mathcal{N}_i = \{\mathbf{q} \in \mathbb{Z}^n : q_i \geq -1 \text{ and } q_j \geq 0\,,\ \text{if } j \neq i\,, \quad j = 1, \ldots, n\}\,,$$

because the factor x_i has been moved out of the sum in (3).

Above we assumed that the origin is a stationary point. Otherwise, we must use shift.

We supposed also that the linear part of the system have been reduced to the diagonal form. Note that there exist general formulas for the Jordan case of the linear part but we do not need to use the general case in this report.

Normal form

The normalization is done with a near-identity transformation:

$$x_i \ = \ z_i + z_i \sum_{\mathbf{q}\in\mathcal{N}_i} h_{i,\mathbf{q}}\mathbf{z}^{\mathbf{q}}, \quad i = 1, \ldots, n \tag{4}$$

after which we have system (3) in the normal form:

$$\dot{z}_i = \lambda_i z_i + z_i \sum_{\substack{\langle \mathbf{q}, \mathbf{L}\rangle = 0 \\ \mathbf{q}\in\mathcal{N}_i}} g_{i,\mathbf{q}}\mathbf{z}^{\mathbf{q}}, \quad i = 1, \ldots, n, \tag{5}$$

where $\mathbf{L} = \{\lambda_1, \ldots, \lambda_n\}$ is the vector of eigenvalues.

Theorem (Bruno 1971)

There exists a formal change (4) reducing (3) to its normal form (5).

Resonance terms

- The important difference between (3) and (5) is a restriction on the range of the summation, which is defined by the equation:

$$\langle \mathbf{q}, \mathbf{L} \rangle = \sum_{j=1}^{n} q_j \lambda_j = 0 \,. \tag{6}$$

I.e. the summation in the normal form (5) contains only terms, for which (6) is valid. They are called resonance terms.

- We rewrite below the normalized equation (5) as

$$\dot{z}_i = \lambda_i z_i + z_i g_i(Z), \tag{7}$$

where $g_i(Z)$ is the re-designate sum.

Note, if the eigenvalues are not comparable then condition (6) is never valid at any components of the vector **q**, because they are integer.

For example such situation takes place if $\lambda_1 = 1, \lambda_2 = \sqrt{2}$. In that case the equation (6) has no solutions, $g_i(Z) = 0$ and the normal form (5), (7) will be a linear system.

Phase Space near Stationary Points

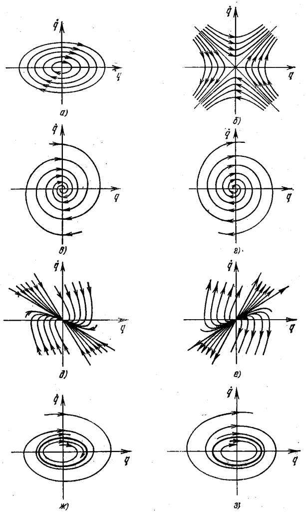


Рис. 17.17. Особые точки на фазовой плоскости; *а*) центр, *б*) седло; *в*) фокус (устойчивый). *г*) фокус (неустойчивый); *д*) узел (устойчивый); *е*) узел (неустойчивый); *ж*, *з*) изолированные циклы (устойчивый и неустойчивый). Об устойчивости и неустойчивости см. ниже.

Calculation of the normal form

The h and g coefficients in (4) and (5) are found by using the recurrent formula:

$$g_{i,\mathbf{q}} + \langle \mathbf{q}, \mathbf{L} \rangle \cdot h_{i,\mathbf{q}} = -\sum_{j=1}^{n} \sum_{\substack{\mathbf{p}+\mathbf{r}=\mathbf{q} \\ \mathbf{p},\mathbf{r} \in \bigcup_i \mathcal{N}_i \\ \mathbf{q} \in \mathcal{N}_i}} (p_j + \delta_{ij}) \cdot h_{i,\mathbf{p}} \cdot g_{j,\mathbf{r}} + \tilde{\Phi}_{i,\mathbf{q}} \,, \quad (8)$$

For this calculation we have two programs.

- in LISP [Edneral, Khrustalev 1992]
- in the high-level language of the MATHEMATICA system [Edneral, Khanin 2002].

Conditions **A and** ω

There are two conditions

- Condition **A**. In the normal form (5)

$$g_j(Z) = \lambda_j a(Z) + \bar{\lambda}_j b(Z), \quad j = 1, \dots, n, \tag{9}$$

  where $a(Z)$ and $b(Z)$ are some formal power series.
- Condition ω (on small divisors) [Bruno 1971]. It is fulfilled for almost all vectors **L**. At least it is satisfied at rational eigenvalues.

Theorem (Bruno 1971)

If vector **L** *satisfies Condition* ω *and the normal form* (5) *satisfies Condition* **A** *then the normalizing transformation (4) converges.*

Local integral

Consider the case of a $[N : M]$ resonance in the two-dimension system. The eigenvalues here satisfy the ratio $N \cdot \lambda_1 = -M \cdot \lambda_2$ and from the condition **A** (9) we have

$$g_1(Z) = \lambda_1 a(Z) + \bar{\lambda}_1 b(Z), \quad g_2(Z) = \lambda_2 a(Z) + \bar{\lambda}_2 b(Z),$$

i.e. $N \cdot g_1(Z) = -M \cdot g_2(Z)$.
The normalized system (7) can be conditionally rewritten as

$$N \times \left| \frac{d\ \log(z_1)}{d\ t} = \lambda_1 + g_1(Z)\ , \quad -M \times \right| \frac{d\ log(z_2)}{d\ t} = \lambda_2 + g_2(Z)\ .$$

$$\text{So,} \quad \frac{d\ \log(z_1^N \cdot z_2^M)}{d\ t} = 0 \text{ or } z_1^N \cdot z_2^M = const.$$

It is the first integral.

Near a stationary point the condition **A**:

- Ensures convergence;
- Provides the local integrability;
- Isolates the periodic orbits if the eigenvalues are pure imagine.

  Note, that the local integrability condition above works in the multidimensional case also.

Hypothesis

We are looking for integrability in some domain of the phase space. We will demand local integrability at all relevant stationary points of the system with resonant linear parts to fulfill this.

Note that the condition **A** has been satisfied at all other points, so that local integrability already holds there.

Hypothesis

Local integrability in a neighborhood of each point of some region of the phase space is necessary for the existence of the first integral in this region.

Necessary condition

- The necessary condition of local integrability the condition **A** is an infinite system of algebraic equations in the system parameters.

- Each subsystem of the condition **A** will be the necessary condition of the local integrability also. We can calculate such finite subsystem by the CA program.

- So, we will have the necessary condition of local integrability as a finite system of algebraic equations in the system parameters.

Scheme

Any system is locally integrable in neighborhoods of regular points of the phase space and locally integrable in all cases of stationary points, except of resonant ones. Therefore, our approach requires the study of the local integrability at all stationary points of the region under study only.

- We choose the stationary point;
- We impose restrictions on the parameters of the system, making the chosen stationary point "resonant". If this is not possible, we cannot draw any conclusions about integrability, such point is useless for us.
- At this resonant stationary point we calculate the normal form till some fixed order, create and solve the truncated condition **A** as a system of algebraic equations in the parametric space. If we have several resonant stationary points then we should unite corresponding systems of equations. As alternative we can solve this system for one point and check solutions at other points.
- For found parameter values we try to calculate the global integrals.

Problem

We will illustrate our method on the example of the generalization of the system [Lunkevich, Sibirskii, 1982]

$$\dot{x} = y + 2xy, \quad \dot{y} = -x - x^2 + xy + y^2$$

This system is integrable in semi-plane $x > -1/2$ and not integrable at $x < -1/2$.

It's first integral is

$$I(x, y) = \frac{\left((x+1)^2 - (x+1)y + y^2\right) \exp\left(-\frac{2\arctan\left(\frac{x-2y+1}{\sqrt{3}(x+1)}\right)}{\sqrt{3}}\right)}{2x+1}.$$

Lunkevich, Sibirskii, 1982

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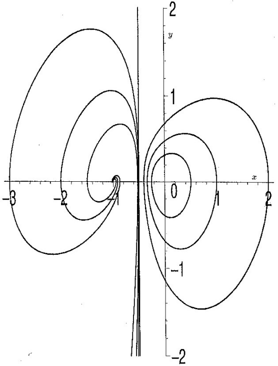


We parametrized this system in the form

$$\begin{aligned} \dot{x} &= \quad y + a_1 x^2 + a_2 xy + a_3 y^2, \\ \dot{y} &= -x + b_1 x^2 + b_2 xy + b_3 y^2. \end{aligned} \tag{10}$$

Here $a_1, a_2, a_3, b_1, b_2, b_3$ are real.

The problem is to obtain restrictions on the parameters from the condition of local integrability at the stationary point in the origin. Then we try to construct the corresponding first integrals.

Condition of the local integrability

We calculated the normal form for the resonance [1:1] by the MATHEMATICA program [Edneral, Khanin, 2002] till the terms of eight order and got the necessary condition of local integrability as four algebraic equations on the parameters. The first two are

$$a_1(a_2 - 2b_1) + a_2a_3 + 2a_3b_3 - b_1b_2 - b_2b_3 = 0,$$

$$4a_1^3(5a_2 - 10b_1 - 9b_3) + a_1^2(a_2(40a_3 - b_2) - 4a_3(b_1 + 5b_3) + 2b_2(9b_1 + 5b_3))+$$
$$a_1(5a_2^3 - a_2^2(9b_1 + 13b_3) + a_2(40a_3^2 + 18a_3b_2 - 18b_1^2 - 8b_1b_3 - b_2^2 - 10b_3^2)+$$
$$20a_3^2(b_1 + 2b_3) + 8a_3b_2(b_1 + b_3) - 40b_1^3 - 40b_1^2b_3 + 9b_1b_2^2 + 20b_1b_3^2 + 13b_2^2b_3+$$
$$36b_3^3) + 5a_2^3a_3 + a_2^2(b_2(b_1 + b_3) - a_3(5b_1 + 9b_3)) + a_2(20a_3^3 + 19a_3^2b_2-$$
$$a_3(10b_1^2 + 8b_1b_3 + b_2^2 + 18b_3^2) + b_2(-19b_1^2 - 18b_1b_3 + b_3^2)) + 40a_3^3b_3+$$
$$10a_3^2b_1b_2 + 18a_3^2b_2b_3 - 20a_3b_1^2b_3 + 5a_3b_1b_2^2 + 4a_3b_1b_3^2 + 9a_3b_2^2b_3 + 40a_3b_3^3-$$
$$20b_1^3b_2 - 40b_1^2b_2b_3 - 5b_1b_2^3 - 40b_1b_2b_3^2 - 5b_2^3b_3 - 20b_2b_3^3 = 0,$$

...

Solutions of the Condition

The MATHEMATICA-11 solver gives 3 subsolutions of the first equation

$$a_1(a_2 - 2b_1) + a_2 a_3 + 2a_3 b_3 - b_1 b_2 - b_2 b_3 = 0.$$

They are

$$\{a_1 \to -b_2/2, a_3 \to b_2/2\},$$

$$\{a_3 \to b_2/2, b_1 \to a_2/2\},$$

$$\{b_3 \to \frac{-a_1 a_2 + 2a_1 b_1 - a_2 a_3 + b_1 b_2}{2a_3 - b_2}\}.$$

We substitute each ot these subsolutions into the remaining 3 equations and solve them by the MATHEMATICA-11 solver.

For example, for the subsolution $\{a_1 \to -b_2/2, a_3 \to b_2/2\}$ we get

$$\begin{array}{l}\{a_1 \to 0\}, \{b_3 \to -\frac{a_2}{2}\}, \{b_3 \to -b_1\},\\ \{a_2 \to -2b_3, b_1 \to -b_3\},\\ \{a_1 \to -\frac{1}{5}\sqrt{-2a_2^2 + 7a_2 b_3 - 3b_3^2}, b_1 \to \frac{1}{5}(a_2 - 3b_3)\},\\ \{a_1 \to \frac{1}{5}\sqrt{-2a_2^2 + 7a_2 b_3 - 3b_3^2}, b_1 \to \frac{1}{5}(a_2 - 3b_3)\},\end{array}$$

and so on.

Below we omit the solutions with roots.

The first case above is

$$\{a_1 \to 0, a_1 \to -b_2/2, a_3 \to b_2/2\}.$$

It is equivalent to

$$\{a_1 \to 0, a_3 \to 0, b_2 \to 0\}.$$

The corresponding case of the system (10) is

$$\begin{aligned} \dot{x} &= \quad y + a_2 x\, y, \\ \dot{y} &= -x + b_1 x^2 + b_3 y^2. \end{aligned}$$

It has the first integral

$$\begin{aligned} I(x,y) = \quad & (a_2 x + 1)^{-\frac{2b_3}{a_2}} (b_3 x(2(a_2 + b_1 - b_3) - b_1(a_2 - 2b_3)x) + \\ & b_3(a_2 - b_3)(a_2 - 2b_3)y^2 + a_2 + b_1 - b_3). \end{aligned}$$

a_2, b_1 and b_3 here are arbitrary parameters.

Difference of the integrable case

wxMaxima 20.06.6 [LunkevichNon.wxm]

Файл Правка Вид Поле Maxima Уравнения Алгебра Анализ Упростить Список Графики Численные расчёты Справка

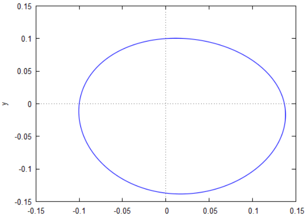
Lunkevich System;
x' = y + a1*x^2 + a2*x*y + a3*y^2
y' =-x + b1*x^2 + b2*x*y + b3*y^2

```
(%i1) load(dynamics);
(%o1) C:/maxima-5.44.0/share/maxima/5.44.0/share/dynamics/dynamics.mac

(%i13) a1 : 1$ a2 : 1$ a3 : 1$ b1 : 1$ b2 : 1$ b3 : 1$
sol : rk([y + a1*x^2 + a2*x*y + a3*y^2,
        -x + b1*x^2 + b2*x*y + b3*y^2],
   [x,y], [0,0.1],[t, 0, 1000, 0.05])$
len : length(sol)$

xg:makelist(sol[k][1],k,1,len)$
yg:makelist(sol[k][2],k,1,len)$
zg:makelist(sol[k][3],k,1,len)$

wxplot2d ([discrete, yg, zg]);
(%t13)
```



Non-integrable case

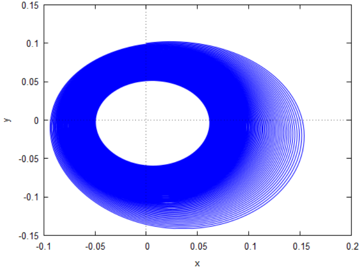
wxMaxima 20.06.6 [LunkevichNon.wxm]

Файл Правка Вид Поле Maxima Уравнения Алгебра Анализ Упростить Список Графики Численные расчёт

(%o13)

```
(%i25) a1 : 1$ a2 : 1$ a3 : 1$ b1 : 1$ b2 : 1$ b3 : 2$
       sol : rk([y + a1*x^2 + a2*x*y + a3*y^2,
                 -x + b1*x^2 + b2*x*y + b3*y^2],
          [x,y], [0,0.1],[t, 0, 1000, 0.05])$
       len : length(sol)$
       xg:makelist(sol[k][1],k,1,len)$
       yg:makelist(sol[k][2],k,1,len)$
       zg:makelist(sol[k][3],k,1,len)$
       wxplot2d ([discrete, yg, zg]);
```

(%t25)



(%o25)

Calculation of the first integrals

An autonomous second order system can be rewritten as a non-autonomous first order equation. Let

$$\frac{d\,x(t)}{d\,t} = P(x(t), y(t)), \quad \frac{d\,y(t)}{d\,t} = Q(x(t), y(t)).$$

We divided the left and right hand sides of the system equations into each other. In result we have the first-order differential equation for $x(y)$ or $y(x)$

$$\frac{d\,x(y)}{d\,y} = \frac{P(x(y), y)}{Q(x(y), y)} \quad \text{or} \quad \frac{d\,y(x)}{d\,x} = \frac{Q(x, y(x))}{P(x, y(x))}.$$

Then we solved them by the MATHEMATICA-11 solver and sometimes got the solution $y(x)$ (or $x(y)$). After that we calculated the integral from this solution by extracting the integration constant $C[1]$.

Results

We have got

1) $$\dot{x} = \quad y + a_2 xy,$$
$$\dot{y} = -x + b_1 x^2 + b_3 y^2,$$
$$I(x,y) = (a_2 x + 1)^{-\frac{2b_3}{a_2}} (b_3 x(2(a_2 + b_1 - b_3) - b_1(a_2 - 2b_3)x) + b_3(a_2 - b_3)(a_2 - 2b_3)y^2 + a_2 + b_1 - b_3);$$

2) $$\dot{x} = \quad y + a_1 x^2 + a_2 xy - a_1 y^2,$$
$$\dot{y} = -x + b_1 x^2 - 2a_1 xy - \tfrac{1}{2} a_2 y^2,$$
$$I(x,y) = -6a_1 x^2 y + 2a_1 y^3 - 3y^2(a_2 x + 1) + x^2(2b_1 x - 3);$$

3) $$\dot{x} = \quad y + a_1 x^2 + a_2 xy - a_1 y^2,$$
$$\dot{y} = -x + \tfrac{1}{2} a_2 x^2 - 2a_1 xy - \tfrac{1}{2} a_2 y^2,$$
$$I(x,y) = x^2(2a_1 y + 1) - \tfrac{2}{3} a_1 y^3 + a_2 x y^2 - \tfrac{1}{3} a_2 x^3 + y^2;$$

4) $\dot{x} = \quad y + a_3 y^2,$
$\dot{y} = -x + 2a_3 xy,$
$I(x,y) = 2a_3 y(a_3 y + 3) + 3\log(1 - 2a_3 y) - 4a_3^2 x^2;$

5) $\dot{x} = \quad y + a_2 xy + \frac{(b_2^2 - a_2^2)y^2}{2b_2},$
$\dot{y} = -x + b_2 xy + \frac{(b_2^2 - a_2^2)y^2}{2a_2},$
$I(x,y) = a_2(\log(-2a_2 b_2 y(a_2 + b_2^2 x) + 2a_2(a_2 + b_2^2 x) +$
$b_2^2(a_2 - b_2)(a_2 + b_2)y^2) + b_2 y) - b_2^2 x;$

6) $\dot{x} = \quad y + a_2 xy - b_2 y^2,$
$\dot{y} = -x - a_2 x^2 + b_2 xy,$
$I(x,y) = x^2 + y^2;$

7) $\dot{x} = \quad y + a_1 x^2 + a_2 xy - a_1 y^2,$
$\dot{y} = -x + b_1 x^2 - 2a_1 xy - b_1 y^2,$

$I(x,y) = \int \frac{x - b_1 x^2 + 2a_1 xy + b_1 y^2}{R(x,y)}\, dx,$ where

$$R(x,y) = -1 - x(-3a_1^2 x + a_2(-1 + b_1 x)^2 + b_1(-2 + x(b_1 + 2a_1^2 x))) + a_1 x(-a_2 + 2b_1 + 6a_1^2 x + 3a_2 b_1 x)y + (b_1(a_2 + b_1)(1 + a_2 x) + a_1^2(3 + 2(a_2 + b_1)x))y^2 - a_1(2a_1^2 + a_2 b_1)y^3.$$

Many thanks to M.V. Demina for this case.

Liénard equation

We will check the method on the example of the Liénard-like equation

$$\ddot{x} = f(x)\dot{x} + g(x), \tag{11}$$

Here we assume that $f(x)$ and $g(x)$ are polynomials. Usually, the Liénard equation assumes that $f(x)$ is an even function and $g(x)$ is an odd function [Lienard 1928]. We do not assume a certain parity for them, so we are talking about the Liénard-type equation.

Many famous equations are partial cases of this one, for example

$$\ddot{x} + x = \varepsilon x^3 \quad – \quad \text{Duffing's equation,}$$

$$\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = 0 \quad – \quad \text{van der Pol's equation.}$$

Van der Pol $\varepsilon > 0$

Van-der-Pol's equation
x" + e (x^2-1) x' + x = 0,

x' = y,
y' =-x - e (x^2-1) y.

(%i1) load(dynamics);
(%o1) C:/maxima-5.44.0/share/maxima/5.44.0/share/

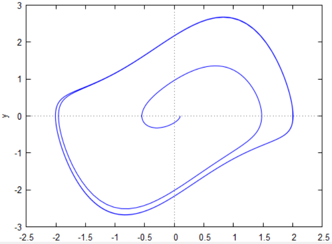
(%i8)
```
e : 1;
sol : rk([y,  - x - e*(x^2-1)*y],
         [x,y],  [0.1, 0.], [t, 0., 100., 0.05])$

len : length(sol)$
xg:makelist(sol[k][1],k,1,len)$
yg:makelist(sol[k][2],k,1,len)$
zg:makelist(sol[k][3],k,1,len)$

wxplot2d ([discrete, yg, zg]);
```

(%o2) 1

(%t8)

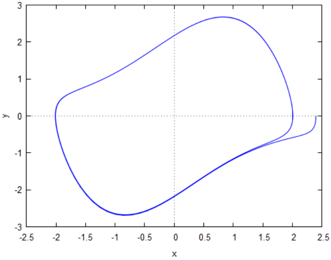


Van der Pol $\varepsilon > 0$

(%i15)
```
a : 1;
sol : rk([y,  - x - a*(x^2-1)*y],
          [x,y],  [2.4, 0.], [t, 0., 100., 0.05])$
len : length(sol)$
xg:makelist(sol[k][1],k,1,len)$
yg:makelist(sol[k][2],k,1,len)$
zg:makelist(sol[k][3],k,1,len)$
wxplot2d ([discrete, yg, zg]);
```

(%o9) 1

(%t15)

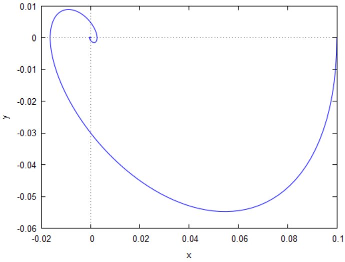


Van der Pol $\varepsilon < 0$

(%i22)
```
a : -1;
sol : rk([y,  - x - a*(x^2-1)*y],
          [x,y],  [0.1, 0.], [t, 0., 100., 0.05])$
len : length(sol)$
xg:makelist(sol[k][1],k,1,len)$
yg:makelist(sol[k][2],k,1,len)$
zg:makelist(sol[k][3],k,1,len)$
wxplot2d ([discrete, yg, zg]);
```

(%o16) -1

(%t22)



We parametrize equation (11) as the dynamical system

$$\begin{cases} \dot{x} = y \\ \dot{y} = (a_0 + a_1 x)\, y + b_1 x + b_2 x^2 + b_3 x^3 \end{cases}, \tag{12}$$

here parameters a_0, a_1, b_1, b_2, b_3 are real.

Resonant Case of the System

The simplest (but not the only one) possibility to satisfy the resonance conditions is to fix the parameter b_1 in system (12).
The matrix and of the linear part of system (12) and its eigenvalues $\lambda_{1,2}$ are

$$\begin{pmatrix} 0 & 1 \\ b_1 & a_0 \end{pmatrix}, \quad \begin{cases} \lambda_1 = \frac{1}{2}\left(a_0 - \sqrt{a_0^2 + 4b_1}\right) \\ \lambda_2 = \frac{1}{2}\left(a_0 + \sqrt{a_0^2 + 4b_1}\right) \end{cases} . \qquad (13)$$

The condition of the resonance $(1 : M)$ is $\lambda_1 = -M \cdot \lambda_2$. It is

$$\begin{cases} a_0 = 0, & \text{if } M = 1 \\ b_1 = \frac{{a_0}^2 M}{(M-1)^2}, & \text{if } M \neq 1 \end{cases} .$$

So, we should consider the system

$$\left\{ \begin{array}{l} \dot{x} = y \\ \dot{y} = a_1 x\, y + b_1 x + b_2 x^2 + b_3 x^3 \end{array} \right., \qquad \text{if } M = 1,$$

or

$$\left\{ \begin{array}{l} \dot{x} = y \\ \dot{y} = (a_0 + a_1 x)\, y + \frac{{a_0}^2 M}{(M-1)^2} x + b_2 x^2 + b_3 x^3 \end{array} \right., \quad \text{if } M \neq 1. \tag{14}$$

Conditions of the Local Integrability at $M = 1$

We calculated the normal form for the system (14) at $M = 1$ till the 8^{th} order and get the condition **A** as the system of four equations in the parameters of the problem (12)

$$
\begin{aligned}
& a_1 b_1^9 b_2 = 0, \\
& 51840 a_1 b_1^6 b_2 (5a_1^2 b_1 + 39 b_1 b_3 - 20 b_2^2) = 0, \\
& 90 a_1 b_1^3 b_2 (2197 a_1^4 b_1^2 + a_1^2 b_1 (31962 b_1 b_3 - 18881 b_2^2) + 113157 b_1^2 b_3^2 - 159348 b_1 b_2^2 b_3 + \\
& \quad 41236 b_2^4) = 0, \\
& a_1 b_2 (165079 a_1^6 b_1^3 + 3 a_1^4 b_1^2 (1182079 b_1 b_3 - 403371 b_2^2) + 3 a_1^2 b_1 (8222823 b_1^2 b_3^2 - \\
& \quad 8332361 b_1 b_2^2 b_3 + 814814 b_2^4) + 54802575 b_1^3 b_3^3 - 128936610 b_1^2 b_2^2 b_3^2 + 62067540 b_1 b_2^4 b_3 - \\
& \quad 7371160 b_2^6) = 0.
\end{aligned}
\tag{15}
$$

Note that adding equations from higher orders of the normal form can not increase the set of solutions.

The solver of the MATHEMATICA-11 system gives for (15) two solutions $a_1 \to 0$ or $b_2 \to 0$. The corresponding equations are

$$\begin{array}{ll} \alpha) & \ddot{x} = b_1 x + b_2 x^2 + b_3 x^3, \\ \beta) & \ddot{x} = a_1 x\, \dot{x} + b_1 x + b_3 x^3. \end{array}$$

These both cases are integrable. The corresponded first integrals are in Appendix A.

We calculated the normal form for resonances (1:2), (1:3), and (1:4) till the terms of ninth, twelfth and fifteenth orders. For each case, we have obtained the algebraic systems in the parametric space with three equations for the each resonance.

These three equations for $M = 2$ are

$$\begin{gathered}
a_0(13a_0a_1b_2^2 - 11b_2^3 + a_0^2b_2(26a_1^2 + 43b_3) + a_0^3(2a_1^3 - 29a_1b_3)) = 0, \\
a_0(43628a_0a_1b_2^5 + 95392b_2^6 - 4a_0^3a_1b_2^3(85609a_1^2 + 47372b_3) - a_0^2b_2^4(349006a_1^2 + 932317b_3) + 2a_0^5a_1b_2(45454a_1^4 + 524839a_1^2b_3 + 56431b_3^2) + a_0^4b_2^2(39206a_1^4 + 1900239a_1^2b_3 + 2097973b_3^2) + a_0^6(6508a_1^6 - 56432a_1^4b_3 - 1187243a_1^2b_3^2 - 853200b_3^3)) = 0, \\
5173440108a_0a_1b_2^8 + 955445018b_2^9 + 3a_0^2b_2^7(698320089a_1^2 - 9972352909b_3) - 7a_0^3a_1b_2^6(1748728465a_1^2 + 11316860901b_3) + 12a_0^5a_1b_2^4(991433401a_1^4 + 17016366014a_1^2b_3 + 26519945973b_3^2) + 6a_0^4b_2^5(-1267606169a_1^4 + 6907374970a_1^2b_3 + 33195427246b_3^2) + a_0^6b_2^3(10471055755a_1^6 + 86771145246a_1^4b_3 - 281583701214a_1^2b_3^2 - 445056171871b_3^3) - 3a_0^7a_1b_2^2(58777497a_1^6 + 18754567420a_1^4b_3 + 185085944446a_1^2b_3^2 + 149954547727b_3^3) - 3a_0^8b_2(453029342a_1^8 + 9954001713a_1^6b_3 + 43938980394a_1^4b_3^2 - 82884194501a_1^2b_3^3 - 94237137000b_3^4) + a_0^9(-96473174a_1^9 - 61382001a_1^7b_3 + 25992267942a_1^5b_3^2 + 177095320637a_1^3b_3^3 + 108865809000a_1b_3^4) = 0.
\end{gathered} \tag{16}$$

Case $M = 2$

The all solutions of this algebraic system which were calculated by the MATHEMATICA-11 solver at the resonant condition $b_1 \to 2a_0^2$ for $M = 2$ are

$$\begin{array}{ll} a) & \{a_0 \to 0, b_2 \to 0\}, \\ b) & \{b_2 \to -a_0 a_1, b_3 \to 0\}, \\ c) & \{b_2 \to -\frac{4}{7} a_0 a_1, b_3 \to -\frac{6}{49}, a_1^2\}, \\ d) & \{b_2 \to -\frac{1}{3} a_0 a_1, b_3 \to -\frac{1}{9} a_1^2\}, \\ e) & \{b_2 \to 3 a_0 a_1, b_3 \to a_1^2\}, \\ f) & \{a_1 \to 0, b_2 \to 0, b_3 \to 0\}. \end{array} \tag{17}$$

The system (14) has 2 stationary points in addition to the origin. At all sets of parameters above we checked the integrability condition at the all other stationary points. At all cases except of d) eigenvalues are not comparable. At case d) there are two additional resonant stationary points with pure imagine eigenvalues. We check validity of the condition **A** here till the 21th order of the normal form. So for cases a) – f) the condition of Hypotheses are satisfied.

The corresponding equations are

$$
\begin{array}{ll}
a) & \ddot{x} = a_1 x\, \dot{x} + b_3 x^3, \\
b) & \ddot{x} = (a_0 + a_1 x)\, \dot{x} + 2{a_0}^2 x - a_0 a_1 x^2, \\
c) & \ddot{x} = (a_0 + a_1 x)\, \dot{x} + 2{a_0}^2 x - \frac{4}{7} a_0 a_1 x^2 - \frac{1}{49} 6{a_1}^2 x^3, \\
d) & \ddot{x} = (a_0 + a_1 x)\, \dot{x} + 2{a_0}^2 x - \frac{1}{3} a_0 a_1 x^2 - \frac{1}{9} {a_1}^2 x^3, \\
e) & \ddot{x} = (a_0 + a_1 x)\, \dot{x} + 2{a_0}^2 x + 3 a_0 a_1 x^2 + {a_1}^2 x^3, \\
f) & \ddot{x} = a_0 \dot{x} + 2a_0^2 x.
\end{array}
\qquad (18)
$$

All of them are integrable. The first integrals are in the Appendix A. The case a) corresponds to the nilpotent matrix of the linear part of system (14) with zero eigenvalues and f) corresponds to a linear system. a), d), f) are partial cases of 2.2.3-2. in the collection of the solvable equations [Polyanin, Zaitsev 2003].

Case $M = 3$

The systems of equations which was created for the resonance (1:3) with $\{b_1 \to 3a_0^2/4\}$ is recorded in Appendix B.The corresponding ODEs are

$$
\begin{aligned}
a') &\quad \ddot{x} = a_1 x\,\dot{x} + b_3 x^3,\\
b') &\quad \ddot{x} = (a_0 + a_1 x)\,\dot{x} + \tfrac{3}{4} {a_0}^2 x - 2 a_0 a_1 x^2 + a_1^2 x^3,\\
c') &\quad \ddot{x} = (a_0 + a_1 x)\,\dot{x} + \tfrac{3}{4} {a_0}^2 x - \tfrac{5}{3} a_0 a_1 x^2 + a_1^2 x^3,\\
d') &\quad \ddot{x} = (a_0 + a_1 x)\,\dot{x} + \tfrac{3}{4} {a_0}^2 x - \tfrac{3}{4} a_0 a_1 x^2 - 6/49 a_1^2 x^3,\\
e') &\quad \ddot{x} = (a_0 + a_1 x)\,\dot{x} + \tfrac{3}{4} {a_0}^2 x - \tfrac{6}{11} a_0 a_1 x^2 - 15/121 a_1^2 x^3,\\
f') &\quad \ddot{x} = (a_0 + a_1 x)\,\dot{x} + \tfrac{3}{4} {a_0}^2 x - \tfrac{1}{3} a_0 a_1 x^2 - 1/9 a_1^2 x^3,\\
g') &\quad \ddot{x} = (a_0 + a_1 x)\,\dot{x} + \tfrac{3}{4} {a_0}^2 x + \tfrac{9}{8} a_0 a_1 x^2 + 3/8 a_1^2 x^3,\\
h') &\quad \ddot{x} = 3/4 {a_0}^2 x + a_0 y.
\end{aligned}
$$

All these case are integrable. The first integral are in Appendix A. Note, the number of solutions increases with the resonance order because increases the equations degree. The cases a), a') and f), h') are the same.

Note, the cases d), f') (and really at the higher resonances) are the same. In a heuristic tradition we can suppose that b_1 does not depend on the resonance order, i.e. the parameter b_1 in d) can be voluntary

$$\ddot{x} = (a_0 + a_1 x)\,\dot{x} + b_1 x - \frac{1}{3} a_0 a_1 x^2 - \frac{1}{9} {a_1}^2 x^3.$$

This case is exactly up to a multiplier the example 2.2.3-2.4 from book [Polyanin, Zaitsev 2003] at $a_1 \to 3a, a_0 \to b$. It is indeed integrable at voluntary b_1. The corresponding first integral is

$$\left[\frac{x(-3\sqrt{a_0^2+4b_1}+3a_0+2a_1 x)-6y}{3(\sqrt{a_0^2+4b_1}+a_0)x+2a_1 x^2-6y}\right]^{a_0} \times \left[\frac{(6a_0 x-6(\frac{3b_1}{a_1}+y)+2a_1 x^2)^2}{3a_0 a_1 x^3-9a_0 xy+a_1^2 x^4-3x^2(2a_1 y+3b_1)+9y^2}\right]^{\sqrt{a\ldots}}$$

Other example

We treated the degenerated system [Bruno, Edneral, Romanovski 2017]

$$dx/dt = -y^3 - b\,x^3 y + a_0\,x^5 + a_1\,x^2 y^2,$$
$$dy/dt = (1/b)\,x^2 y^2 + x^5 + b_0\,x^4 y + b_1\,x\,y^3,$$

with five arbitrary real parameters $b \neq 0, a_1, a_2, b_1, b_2$.

With this technique, we found seven two-dimensional conditions at which the system above is integrable

$$
\begin{array}{ll}
1)\ b_1 = 0, & a_0 = 0, a_1 = -b_0\, b, \;\; b^2 \neq 2/3; \\
2)\ b_1 = -2\, a_1, & a_0 = a_1\, b, b_0 = b_1\, b, \; b^2 \neq 2/3; \\
3)\ b_1 = 3/2\, a_1, & a_0 = a_1\, b, b_0 = b_1\, b, \; b^2 \neq 2/3; \\
4)\ b_1 = 8/3\, a_1, & a_0 = a_1\, b, b_0 = b_1\, b, \; b^2 \neq 2/3; \\
5)\ b_1 = 3/2\, a_1, & a_0 = (2b_0 + b\,(3a_1 - 2b_1))/3, \; b = \sqrt{2/3}; \\
6)\ b_1 = 6\, a_1 + 2\sqrt{6} b_0, & a_0 = (2b_0 + b\,(3a_1 - 2b_1))/3, \; b = \sqrt{2/3}; \\
7)\ b_1 = -2/3\, a_1, & 20a_0 + 2\sqrt{6}a_1 + 4b_0 + 3\sqrt{6}b_1 = 0, \\
 & 3a_0 - 2b_0 \neq b\,(3a_1 - 2b_1), \qquad b = \sqrt{2/3}.
\end{array}
$$

For each of these conditions, we found the first integral of motion.

Conclusions

- The hypothesis is proposed that local integrability in a neighborhood of each point of some domain of the phase space leads to the existence of the first integral in this domain.
- This hypothesis allows to propose the algorithmic scheme for searching integrable cases. This scheme is not limited to two dimensions.
- For the two-dimensional autonomous polynomial systems we found several sets of parameters at which they have the first integrals and, in this way solvable.

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Many thanks for your attention